Radial Löwner equation and Dispersionless cmKP Hierarchy

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Abstract

It has been shown that the dispersionless KP hierarchy (or the Benney hierarchy) is reduced to the chordal Löwner equation. We show that the radial Löwner equation also gives reduction of a dispersionless type integrable system. The resulting system acquires another degree of freedom and becomes the dcmKP hierarchy, which is a "half" of the dispersionless Toda hierarchy.

Key words: radial Löwner equation, dcmKP hierarchy.

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1 Introduction

Recently reductions and hodograph solutions of dispersionless and/or Whitham type integrable systems are intensively studied [GTs, YG, MMAM, GMMA]. In this article we report another example — reduction of the dispersionless coupled modified KP (dcmKP) hierarchy to the (radial) Löwner equation.

The Löwner equation

$$\frac{\partial g}{\partial \lambda}(\lambda, z) = g(\lambda, z)\frac{\kappa(\lambda) + g(\lambda, z)}{\kappa(\lambda) - g(\lambda, z)}\frac{\partial \phi(\lambda)}{\partial \lambda} \tag{1}$$

was introduced by K. Löwner [L] in an attempt to solve the Bieberbach conjecture. (See, e.g., [D], Chapter 3.) It is an evolution equation of the conformal mapping $g(\lambda, z)$ (as a function of z) from a chain of subdomains of the complement of the unit disk onto the complement of the unit disk, normalized as

$$g(\lambda, z) = e^{-\phi(\lambda)}z + b_0(\lambda) + b_1(\lambda)z^{-1} + b_2(\lambda)z^{-2} + \cdots .$$

Namely, g is normalized so that it maps a fixed interior point of the domain ($z = \infty$) to a fixed interior point ($z = \infty$).

We can also define the same kind of equation with different normalization which is called the "*chordal* Löwner equation":

$$\frac{\partial g}{\partial \lambda}(\lambda, z) = \frac{1}{g(\lambda, z) - U_i(\lambda)}\frac{\partial a_1(\lambda)}{\partial \lambda}. \tag{2}$$

Here g is a conformal mapping from a subdomain of the upper half plain to the upper half plain, normalized as

$$g(\lambda, z) = z + a_1(\lambda)z^{-1} + a_2(\lambda)z^{-2} + \cdots .$$

Hence this maps a fixed boundary point ($z = \infty$) to a fixed boundary point ($z = \infty$). See [LSW] §2.3 for details. The original Löwner equation is, therefore, often called the "*radial* Löwner equation".

The reduction of the dispersionless KP hierarchy [dKP, TT2, TT3] to the chordal Löwner equation (and its generalization) has been studied by Gibbons and Tsarev [GTs], Yu and Gibbons [YG], Mañas, Martínez Alonso and Medina [MMAM] and others. Our question is: what about the radial Löwner equation? Note that the above mentioned works on the chordal case does not contain the radial Löwner equation because of the normalization at the infinity.

The answer is that there appears another degree of freedom and the resulting system turns out to be the dcmKP hierarchy. The dcmKP hierarchy introduced by Teo [Te1] is an extension of the dispersionless mKP hierarchy [Ta] with an additional degree of freedom, or in other words, a "half" of the dispersionless Toda lattice hierarchy [TT1, TT3].

Mañas, Martínez Alonso and Medina [MMAM] generalized the chordal Löwner equation (2) by changing the rational function in the right hand side. The radial Löwner equation is also generalized in the same way. In the theory of univalent functions those generalized

equations are called the Löwner-Kufarev equations. We show that a certain class of the Löwner-Kufarev equations is transformed to the (radial) Löwner equation with respect to the Riemann invariants. This class of Löwner-Kufarev equations has already been considered in [PV] and [C]. Prokhorov and Vasil'ev showed in [PV] that such class is related to an integrable system, which is presumably different from ours. Cardy introduced a stochastic version in [C].

In the following two sections we review the two ingredients, the Löwner equation and the dcmKP hierarchy. The main result is presented in §4. The reduction of the Löwner-Kufarev equation is explained in §6.

The results of this article was announced in [TT4].

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2 Radial Löwner equation

In this section we review the (radial) Löwner equation and introduce related notions. Since we are interested in algebro-analytic nature of the system, we omit reality/positivity conditions which are essential in the context of the complex analysis.

The Löwner equation is a system of differential equations for a function

$$w = g(\boldsymbol{\lambda}, z) = e^{-\phi(\boldsymbol{\lambda})} z + b_0(\boldsymbol{\lambda}) + b_1(\boldsymbol{\lambda}) z^{-1} + b_2(\boldsymbol{\lambda}) z^{-2} + \cdots \tag{3}$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ and z are independent variables. In the complex analysis the variable z moves in a subdomain of the complement of the unit disk and the variables λ_i parametrize the subdomain. In our context $g(\boldsymbol{\lambda}, z)$ is considered as a generating function of the unknown functions $\phi(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$. We assume that for each $i = 1, \dots, N$ a *driving function* $\kappa_i(\boldsymbol{\lambda})$ is given. The *Löwner equation* is the following system:

$$\frac{\partial g}{\partial \lambda_i}(\boldsymbol{\lambda}, z) = g(\boldsymbol{\lambda}, z) \frac{\kappa_i(\boldsymbol{\lambda}) + g(\boldsymbol{\lambda}, z)}{\kappa_i(\boldsymbol{\lambda}) - g(\boldsymbol{\lambda}, z)} \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i}, \qquad i = 1, \dots, N. \tag{4}$$

(The original Löwner equation (1) is the case $N = 1$.)

Later the inverse function of $g(\boldsymbol{\lambda}, z)$ with respect to the z-variable will be more important than g itself. We denote it by $f(\boldsymbol{\lambda}, w)$:

$$z = f(\boldsymbol{\lambda}, w) = e^{\phi(\boldsymbol{\lambda})} w + c_0(\boldsymbol{\lambda}) + c_1(\boldsymbol{\lambda}) w^{-1} + c_2(\boldsymbol{\lambda}) w^{-2} + \cdots . \tag{5}$$

It satisfies $g(\boldsymbol{\lambda}, f(\boldsymbol{\lambda}, w)) = w$ and $f(\boldsymbol{\lambda}, g(\boldsymbol{\lambda}, z)) = z$, from which we can determine the coefficients $c_n(\boldsymbol{\lambda})$ in terms of $\phi(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})$. The Löwner equation (4) is rewritten as the equation for $f(\boldsymbol{\lambda}, w)$ as follows:

$$\frac{\partial f}{\partial \lambda_i}(\boldsymbol{\lambda}, w) = w \frac{w + \kappa_i(\boldsymbol{\lambda})}{w - \kappa_i(\boldsymbol{\lambda})} \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial f}{\partial w}(\boldsymbol{\lambda}, w). \tag{6}$$

This equation leads to the compatibility condition of κ_i's:

$$\frac{\partial \kappa_j}{\partial \lambda_i} = -\kappa_j \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \frac{\partial \phi}{\partial \lambda_i}, \tag{7}$$

$$\frac{\partial^2 \phi}{\partial \lambda_i \partial \lambda_j} = \frac{4 \kappa_i \kappa_j}{(\kappa_i - \kappa_j)^2} \frac{\partial \phi}{\partial \lambda_i} \frac{\partial \phi}{\partial \lambda_j}, \tag{8}$$

for any i, j $(i \neq j)$.

The *Faber polynomials* are defined as follows [Te2]:

$$\Phi_n(\boldsymbol{\lambda}, w) := (f(\boldsymbol{\lambda}, w)^n)_{\geq 0}. \tag{9}$$

Here $(\cdot)_{\geq 0}$ is the truncation of the Laurent series in w to its polynomial part. (See also [D], §4.1.) The generating function of Φ_n's is expressed in terms of $g(\boldsymbol{\lambda}, z)$:

$$-\sum_{n=1}^{\infty} \frac{\Phi_n(w)}{n} z^{-n} = \log \frac{g(z) - w}{e^{-\phi} z} = \log(g(z) - w) + \phi - \log z. \tag{10}$$

See [D] §4.1 or [Te2] §2.2. By differentiating (10), we have generating functions of derivatives of Faber polynomials, which will be useful later:

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial \Phi_n}{\partial w}(\boldsymbol{\lambda}, w) z^{-n} &= \frac{1}{w - g(\boldsymbol{\lambda}, z)}, \\
-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \Phi_n}{\partial \lambda_i \partial w}(\boldsymbol{\lambda}, w) z^{-n} &= \frac{g(\boldsymbol{\lambda}, z)(\kappa_i + g(\boldsymbol{\lambda}, z))}{(g(\boldsymbol{\lambda}, z) - w)^2 (\kappa_i - g(\boldsymbol{\lambda}, z))} \frac{\partial \phi}{\partial \lambda_i}, \\
-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 \Phi_n}{\partial w^2}(\boldsymbol{\lambda}, w) z^{-n} &= -\frac{1}{(w - g(\boldsymbol{\lambda}, z))^2}.
\end{aligned} \tag{11}$$

The second equation is a consequence of the Löwner equation (4).

3 dcmKP hierarchy

We give a formulation of the dcmKP hierarchy different from Teo's [Te1]. The equivalence (up to a gauge factor) will be explained in a forthcoming paper.

The independent variables of the system are $(s,t) = (s, t_1, t_2, \dots)$. The unknown functions $\phi(s,t)$ and $u_n(s,t)$ $(n = 0, 1, 2, \dots)$ are encapsulated in the series

$$\mathcal{L}(s,t;w) = e^{\phi(s,t)} w + u_0(s,t) + u_1(s,t) w^{-1} + u_2(s,t) w^{-2} + \cdots, \tag{12}$$

where w is a formal variable. The *dispersionless coupled modified KP hierarchy* (dcmKP hierarchy) is the following system of differential equations:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad n = 1, 2, \dots. \tag{13}$$

Here the Poisson bracket $\{,\}$ is defined by

$$\{f(s,w), g(s,w)\} := w \frac{\partial f}{\partial w} \frac{\partial g}{\partial s} - w \frac{\partial f}{\partial s} \frac{\partial g}{\partial w}, \tag{14}$$

and $\mathcal{B}_n$ is the polynomial in w defined by

$$\mathcal{B}_n := (\mathcal{L}^n)_{>0} + \frac{1}{2} (\mathcal{L}^n)_0, \tag{15}$$

where $(\cdot)_{>0}$ is the positive power part in w and $(\cdot)_0$ is the constant term with respect to w.

We use the following fact: Assume that there exists a function $S(z,s,t)$ with the expansion

$$S(z,s,t) = \sum_{n=1}^{\infty} t_n z^n + s \log z - \frac{1}{2}\varphi - \sum_{n=1}^{\infty} \frac{v_n}{n} z^{-n}, \tag{16}$$

where φ and v_n are functions of (s,t). A function $\mathcal{L} = \mathcal{L}(s,t;w)$ of the form (12) is a solution of the dcmKP hierarchy if and only if the following equation holds:

$$dS(\mathcal{L}, s, t) = \mathcal{M}\, d\log \mathcal{L} + \log w\, ds + \sum_{n=1}^{\infty} \mathcal{B}_n dt_n, \tag{17}$$

where $\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n}$ is the Orlov-Schulman function and $\mathcal{B}_n$ is defined by (15). In other words, $\mathcal{L}$ is a solution if and only if

$$\left(\frac{\partial S}{\partial t_n}\right)_{z,s,t_m(m\neq n)\text{fixed}, z=\mathcal{L}} = \mathcal{B}_n \tag{18}$$

and

$$\left(\frac{\partial S}{\partial s}\right)_{z,t_n \text{fixed}, z=\mathcal{L}} = \log w \tag{19}$$

hold. The proof is the same as the dKP or dToda case, [TT1, TT2, TT3]. Namely, we have only to note that equation (17) implies $d\omega = 0$ and $\omega \wedge \omega = 0$ where $\omega := d \log w \wedge ds + \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dt_n$.

Remark 3.1. It follows from the constant term in (18) and (19) that the function $\varphi(s,t)$ satisfies equations (31) below.

As is shown in [Te1], a solution of the dispersionless Toda hierarchy is a solution of the dcmKP hierarchy if the half of the time variables are fixed. In this case the function φ comes from the Toda field.

4 Reduction of dcmKP hierarchy to radial Löwner equation

In this section we show that a specialization of the variables $\boldsymbol{\lambda}$ in $f(\boldsymbol{\lambda}, w)$ gives a solution of the dcmKP hierarchy.

Suppose $\boldsymbol{\lambda}(s,t) = (\lambda_1(s,t), \ldots, \lambda_N(s,t))$ satisfies the equations

$$\frac{\partial \lambda_i}{\partial t_n} = v_i^n(\boldsymbol{\lambda}(s,t)) \frac{\partial \lambda_i}{\partial s}, \qquad i = 1, \ldots, N, \tag{20}$$

where $v_j^n(\boldsymbol{\lambda})$ are defined by

$$v_j^n(\boldsymbol{\lambda}) := \kappa_j(\boldsymbol{\lambda}) \frac{\partial \Phi_n}{\partial w}(\boldsymbol{\lambda}, \kappa_j(\boldsymbol{\lambda})) = \left.\frac{\partial \Phi_n}{\partial \log w}(\boldsymbol{\lambda}, w)\right|_{w=\kappa_j(\boldsymbol{\lambda})}. \tag{21}$$

Equations (20) say that $\lambda_i(s,t)$ are Riemann invariants with characteristic speed v_i^n.

Lemma 4.1. *The functions $v_j^n(\boldsymbol{\lambda})$ satisfy the equations*

$$\frac{\partial v_j^n}{\partial \lambda_i} = V_{ij}(v_i^n - v_j^n), \tag{22}$$

where

$$V_{ij} := \frac{2\kappa_i \kappa_j}{(\kappa_i - \kappa_j)^2} \frac{\partial \phi}{\partial \lambda_i}. \tag{23}$$

Proof. By the chain rule and the generating function expressions (11), we have

$$\begin{aligned} -\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial v_j^n}{\partial \lambda_i} z^{-n} &= \frac{g}{(g-\kappa_j)^2} \left(-\frac{\partial \kappa_j}{\partial \lambda_i} + \frac{\kappa_j (g+\kappa_i)}{\kappa_i - g} \frac{\partial \phi}{\partial \lambda_i} \right) \\ &= \frac{g\kappa_j}{(g-\kappa_j)^2} \frac{\partial \phi}{\partial \lambda_i} \left(\frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} + \frac{g+\kappa_i}{\kappa_i - g} \right) \\ &= -\frac{2\kappa_i \kappa_j}{(\kappa_i - \kappa_j)^2} \frac{\partial \phi}{\partial \lambda_i} \left(\frac{\kappa_i}{g-\kappa_i} - \frac{\kappa_j}{g-\kappa_j} \right) \end{aligned} \tag{24}$$

Here we used (7). The coefficient of z^{-n} gives (22) because of (11). $\square$

The hydrodynamic type equations (20) can be solved by the generalized hodograph method of Tsarev [Ts]:

Proposition 4.2. *(i) For any triple of distinct indices i, j, k, we have*

$$\frac{\partial V_{jk}}{\partial \lambda_i} = \frac{\partial V_{ik}}{\partial \lambda_j}. \tag{25}$$

Hence the system for functions $F_i(\boldsymbol{\lambda})$

$$\frac{\partial F_j}{\partial \lambda_i} = V_{ij}(F_i - F_j) \tag{26}$$

is compatible according to [Ts] §3.

(ii) Let $\{F_i\}_{i=1,\dots,N}$ be a solution of (26). Then the hodograph relation

$$F_i(\boldsymbol{\lambda}(s,t)) = s + \sum_{n=1}^{\infty} v_i^n(\boldsymbol{\lambda}(s,t))\, t_n \tag{27}$$

defines a solution $\boldsymbol{\lambda}(s,t)$ of (20) as an implicit function.

Proof. (i) is a consequence of direct computation. In fact the both hand sides of (26) are equal to

$$\frac{4\kappa_i\kappa_j\kappa_k(\kappa_i+\kappa_j)}{(\kappa_i-\kappa_k)(\kappa_j-\kappa_k)(\kappa_i-\kappa_j)^2}\frac{\partial\phi}{\partial\lambda_i}\frac{\partial\phi}{\partial\lambda_j},$$

due to (7) and (8).

(ii) The idea of the proof is the same as that of Theorem 10 of [Ts]. By differentiating (27) by s and t_n, we obtain

$$\sum_{j=1}^{N}\left(\frac{\partial F_i}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t)) - \sum_{m=1}^{\infty}\frac{\partial v_i^m}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t))\, t_m\right)\frac{\partial \lambda_j}{\partial s} = 1, \tag{28}$$

$$\sum_{j=1}^{N}\left(\frac{\partial F_i}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t)) - \sum_{m=1}^{\infty}\frac{\partial v_i^m}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t))\, t_m\right)\frac{\partial \lambda_j}{\partial t_n} = v_i^n(\boldsymbol{\lambda}(s,t)). \tag{29}$$

The first factor in the left hand side in (28) and (29) with $j \neq i$ is

$$\begin{aligned}
&\frac{\partial F_i}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t)) - \sum_{m=1}^{\infty}\frac{\partial v_i^m}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t))\, t_m \\
=&\frac{\partial F_i}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t)) - V_{ji}\sum_{m=1}^{\infty}\bigl(v_j^m(\boldsymbol{\lambda}(s,t))\, t_m - v_i^m(\boldsymbol{\lambda}(s,t)) t_m\bigr) \\
=&\frac{\partial F_i}{\partial \lambda_j}(\boldsymbol{\lambda}(s,t)) - V_{ji}\bigl(F_j(\boldsymbol{\lambda}(s,t)) - F_i(\boldsymbol{\lambda}(s,t))\bigr) = 0,
\end{aligned}$$

because of (22), (27) and (26). Hence equations (28) and (29) become

$$\left(\frac{\partial F_i}{\partial \lambda_i}(\boldsymbol{\lambda}(s,t)) - \sum_{m=1}^{\infty} \frac{\partial v_i^m}{\partial \lambda_i}(\boldsymbol{\lambda}(s,t))\, t_m\right) \frac{\partial \lambda_i}{\partial s} = 1,$$
$$\left(\frac{\partial F_i}{\partial \lambda_i}(\boldsymbol{\lambda}(s,t)) - \sum_{m=1}^{\infty} \frac{\partial v_i^m}{\partial \lambda_i}(\boldsymbol{\lambda}(s,t))\, t_m\right) \frac{\partial \lambda_i}{\partial t_n} = v_i^n(\boldsymbol{\lambda}(s,t)),$$

the ratio of which gives (20). □

Our main result is the following.

Theorem 4.3. *Let $f(\boldsymbol{\lambda}, w)$ be a solution of the radial Löwner equation* (6) *of the form* (5) *and $\boldsymbol{\lambda}(s,t)$ be a solution of* (20)*. Then the function $\mathcal{L} = \mathcal{L}(s,t;w)$ defined by*

$$\begin{aligned}\mathcal{L}(s,t;w) &:= f(\boldsymbol{\lambda}(s,t), w) \\ &= e^{\phi(\boldsymbol{\lambda}(s,t))} w + c_0(\boldsymbol{\lambda}(s,t)) + c_1(\boldsymbol{\lambda}(s,t)) w^{-1} + c_2(\boldsymbol{\lambda}(s,t)) w^{-2} + \cdots\end{aligned} \tag{30}$$

is a solution of the dcmKP hierarchy (13)*.*

The rest of this section is devoted to the proof of this theorem. We construct the S function (16), following Mañas, Martínez Alonso and Medina [MMAM], but the zero-mode $-\varphi/2$ in (16) should be added separately.

Lemma 4.4. *There exists a function $\varphi(s,t)$ which satisfies*

$$\frac{\partial \varphi}{\partial t_n} = \Phi_n(\boldsymbol{\lambda}(s,t), 0), \qquad \frac{\partial \varphi}{\partial s} = 2\phi(s,t), \tag{31}$$

for all $n = 1, 2, \dots$.

Proof. We have only to check the compatibility of (31), i.e.,

$$\frac{\partial \Phi_n}{\partial t_m}(\boldsymbol{\lambda}(s,t), 0) = \frac{\partial \Phi_m}{\partial t_n}(\boldsymbol{\lambda}(s,t), 0), \qquad \frac{\partial \Phi_n}{\partial s}(\boldsymbol{\lambda}(s,t), 0) = 2\frac{\partial \phi}{\partial t_n}. \tag{32}$$

Thanks to the generating function expression (10), the left hand side of the second equation of (32) is

$$\begin{aligned}&\frac{\partial \Phi_n}{\partial s}(\boldsymbol{\lambda}(s,t), 0) \\ &= -\, n \operatorname{Res} z^{n-1} \frac{\partial}{\partial s} \left(\log g(\boldsymbol{\lambda}(s,t), z) + \phi(\boldsymbol{\lambda}(s,t)) - \log z\right)\, dz \\ &= -\operatorname{Res} \frac{\partial}{\partial s} \log g(\boldsymbol{\lambda}(s,t), z)\, d(z^n) \\ &= \operatorname{Res} z^n\, d\left(\frac{\partial}{\partial s} \log g(\boldsymbol{\lambda}(s,t), z)\right),\end{aligned} \tag{33}$$

by integration by parts. Since the Löwner equation implies

$$\begin{aligned}\frac{\partial}{\partial s}\log g(\boldsymbol{\lambda}(s,t),z) &= \sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial s}\frac{1}{g(\boldsymbol{\lambda}(s,t),z)}\frac{\partial g}{\partial\lambda_i}\\ &= \sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial s}\frac{\kappa_i+g}{\kappa_i-g}\frac{\partial\phi}{\partial\lambda_i},\end{aligned}\tag{34}$$

we can rewrite (33) by the coordinate transformation $w=g(\boldsymbol{\lambda},z)$ (i.e., $z=f(\boldsymbol{\lambda},w)$) as follows:

$$\begin{aligned}&\frac{\partial\Phi_n}{\partial s}(\boldsymbol{\lambda}(s,t),0)\\ &=\operatorname{Res} f(\boldsymbol{\lambda}(s,t),w)^n\,\frac{\partial}{\partial w}\left(\frac{\partial}{\partial s}\log g(\boldsymbol{\lambda}(s,t),z)\bigg|_{z=f(\boldsymbol{\lambda}(s,t),w)}\right)\,dw\\ &=\sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial s}\operatorname{Res}\left(f(\boldsymbol{\lambda}(s,t),w)^n\frac{\partial}{\partial w}\frac{\kappa_i+w}{\kappa_i-w}\frac{\partial\phi}{\partial\lambda_i}\right)\,dw\\ &=-\sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial s}\frac{\partial\phi}{\partial\lambda_i}\operatorname{Res}\left(f(\boldsymbol{\lambda}(s,t),w)^n\frac{-2\kappa_i}{(w-\kappa_i)^2}\right)\,dw.\end{aligned}\tag{35}$$

The argument which proves (2.5) of [Te2] shows that the residue in the last line is $-2\kappa_i\dfrac{\partial\Phi_n}{\partial w}(\boldsymbol{\lambda}(s,t),\kappa_i)=-2v_i^n$. Hence it follows from (35) that

$$\frac{\partial\Phi_n}{\partial s}(\boldsymbol{\lambda}(s,t),0)=2\sum_{i=1}^{N}v_i^n\frac{\partial\lambda_i}{\partial s}\frac{\partial\phi}{\partial\lambda_i}=2\sum_{i=1}^{N}\frac{\partial\lambda_i}{\partial t_n}\frac{\partial\phi}{\partial\lambda_i}=2\frac{\partial\phi}{\partial t_n},$$

which proves the second equation of (32). The first equation of (32) is proved in the same manner. □

Using this $\varphi(s,t)$, we define the S function as

$$S(z,s,t):=\mathcal{S}(g(\boldsymbol{\lambda}(s,t),z),\boldsymbol{\lambda}(s,t),s,t)-\frac{1}{2}\varphi(s,t),\tag{36}$$

$$\mathcal{S}(w,\boldsymbol{\lambda},s,t):=\mathcal{S}_+(w,\boldsymbol{\lambda},s,t)+\mathcal{S}_-(w,\boldsymbol{\lambda}),\tag{37}$$

where

$$\begin{aligned}\mathcal{S}_+(w,\boldsymbol{\lambda},s,t)&:=\sum_{n=1}^{\infty}t_n\Phi_n(\boldsymbol{\lambda},w)+s\log e^{\phi(\boldsymbol{\lambda})}w\\ &=\sum_{n=1}^{\infty}t_n\Phi_n(\boldsymbol{\lambda},w)+s(\phi(\boldsymbol{\lambda})+\log w),\end{aligned}\tag{38}$$

and $\mathcal{S}_-$ is a power series of w^{-1} without a constant term (i.e., $\mathcal{S}_-(w,\boldsymbol{\lambda}) = O(w^{-1})$) which satisfies the differential equation:

$$\frac{\partial \mathcal{S}_-}{\partial \lambda_i} - w\frac{w+\kappa_i}{w-\kappa_i}\frac{\partial \phi}{\partial \lambda_i}\frac{\partial \mathcal{S}_-}{\partial w} = \frac{2\kappa_i F_i}{w-\kappa_i}\frac{\partial \phi}{\partial \lambda_i}. \tag{39}$$

The compatibility of (39) is ensured by (26). (cf. [GMMA] Proposition 2.)

Let us check that the S function satisfies (18) and (19). The argument is similar to the proof of Proposition 2 of [GMMA]. First we prove (18). By the definition (36), we have

$$\begin{aligned}\frac{\partial S}{\partial t_n} = &\frac{\partial \mathcal{S}}{\partial t_n}(w,\boldsymbol{\lambda},s,t)\Big|_{w=g(\boldsymbol{\lambda}(s,t),z),\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)} \\ &+\sum_{i=1}^{N}\frac{\partial}{\partial \lambda_i}\mathcal{S}(g(\boldsymbol{\lambda},z),\boldsymbol{\lambda},s,t)\Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)}\frac{\partial \lambda_i}{\partial t_n} - \frac{1}{2}\frac{\partial \varphi}{\partial t_n}.\end{aligned} \tag{40}$$

The first term of the right hand side can be written as

$$\begin{aligned}\frac{\partial \mathcal{S}}{\partial t_n}(w,\boldsymbol{\lambda},s,t)\Big|_{w=g(\boldsymbol{\lambda}(s,t),z),\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)} &= \Phi_n(\boldsymbol{\lambda}(s,t), g(\boldsymbol{\lambda}(s,t),z)) \\ &= \Phi_n(\boldsymbol{\lambda}(s,t),w)\end{aligned} \tag{41}$$

by the definitions (37) and (38). We shall show that

$$\frac{\partial}{\partial \lambda_i}\mathcal{S}(g(\boldsymbol{\lambda},z),\boldsymbol{\lambda},s,t)\Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)} = 0, \tag{42}$$

which proves (18) thanks to (31).

We divide (42) into two parts, namely,

$$\left(\frac{\partial}{\partial \lambda_i}\mathcal{S}(g(\boldsymbol{\lambda},z),\boldsymbol{\lambda},s,t)\Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)}\right)_{\geq 0} = 0, \tag{43}$$

and

$$\left(\frac{\partial}{\partial \lambda_i}\mathcal{S}(g(\boldsymbol{\lambda},z),\boldsymbol{\lambda},s,t)\Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)}\right)_{< 0} = 0. \tag{44}$$

We first prove (43). Since $\mathcal{S}_-$ consists of negative powers of w, it suffices to show

$$\left(\frac{\partial}{\partial \lambda_i}\mathcal{S}_+(g(\boldsymbol{\lambda},z),\boldsymbol{\lambda},s,t)\Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)}\right)_{\geq 0} = 0. \tag{45}$$

Recall that we have

$$\Phi_n(\boldsymbol{\lambda}, g(\boldsymbol{\lambda},z)) = z^n + O(z^{-1}), \qquad \log e^{\phi} g(\boldsymbol{\lambda},z) = \log z + O(z^{-1}),$$

by the definition of the Faber polynomials (9) and the normalization (3) of $g(\boldsymbol{\lambda}, z)$. Therefore

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda_i} \mathcal{S}_+(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t) \\
&= \sum_{n=1}^{\infty} t_n \frac{\partial \Phi_n}{\partial \lambda_i}(\boldsymbol{\lambda}, g(\boldsymbol{\lambda}, z)) + s \frac{\partial}{\partial \lambda_i} \log e^{\phi} g(\boldsymbol{\lambda}, z) = O(z^{-1}),
\end{aligned}
$$

and hence the expression in the parentheses in (45) does not contain non-negative powers of w, when z is substituted by $f(\boldsymbol{\lambda}, w)$.

Next we prove (44). From the definition of $\mathcal{S}_+$ (38) we can rewrite the left hand side of (44) as

$$
\begin{aligned}
&\left(\frac{\partial}{\partial \lambda_i} \mathcal{S}(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t) \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)} \right)_{<0} \\
&= \left(\frac{\partial \mathcal{S}}{\partial w}(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t) \frac{\partial g}{\partial \lambda_i}(\boldsymbol{\lambda}, z) \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)} \right)_{<0} \\
&\quad + \frac{\partial \mathcal{S}_-}{\partial \lambda_i}(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t) \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(s,t)}
\end{aligned}
\qquad (46)
$$

Using the Löwner equation, we have

$$
\begin{aligned}
&\frac{\partial \mathcal{S}}{\partial w}(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t) \frac{\partial g}{\partial \lambda_i}(\boldsymbol{\lambda}, z) \\
&= \frac{\partial \mathcal{S}}{\partial w}(g(\boldsymbol{\lambda}, z), \boldsymbol{\lambda}, s, t)\, g(\boldsymbol{\lambda}, z) \frac{\kappa_i(\boldsymbol{\lambda}) + g(\boldsymbol{\lambda}, z)}{\kappa_i(\boldsymbol{\lambda}) - g(\boldsymbol{\lambda}, z)} \frac{\partial \phi}{\partial \lambda_i} \\
&= \left(w \frac{\partial \mathcal{S}_+}{\partial w}(w, \boldsymbol{\lambda}, s, t) - F_i(\boldsymbol{\lambda}) \right) \frac{\kappa_i(\boldsymbol{\lambda}) + w}{\kappa_i(\boldsymbol{\lambda}) - w} \frac{\partial \phi}{\partial \lambda_i} \Big|_{w=g(\boldsymbol{\lambda}, z)}. \\
&\quad + \left(w \frac{\partial \mathcal{S}_-}{\partial w}(w, \boldsymbol{\lambda}, s, t) + F_i(\boldsymbol{\lambda}) \right) \frac{\kappa_i(\boldsymbol{\lambda}) + w}{\kappa_i(\boldsymbol{\lambda}) - w} \frac{\partial \phi}{\partial \lambda_i} \Big|_{w=g(\boldsymbol{\lambda}, z)}.
\end{aligned}
\qquad (47)
$$

Note that

$$
\begin{aligned}
&w \frac{\partial \mathcal{S}_+}{\partial w}(w, \boldsymbol{\lambda}(s,t), s, t) - F_i(\boldsymbol{\lambda}(s,t)) \\
&\qquad = s + \sum_{n=1}^{\infty} t_n w \frac{\partial \Phi_n}{\partial w}(w; \boldsymbol{\lambda}(s,t), s, t) - F_i(\boldsymbol{\lambda}(s,t)),
\end{aligned}
\qquad (48)
$$

which vanishes at $w = \kappa_i(\boldsymbol{\lambda}(s,t))$ because of the hodograph relation (27). Hence

$$
\left(w \frac{\partial \mathcal{S}_+}{\partial w}(w, \boldsymbol{\lambda}(s,t), s, t) - F_i(\boldsymbol{\lambda}(s,t)) \right) \frac{\kappa_i(\boldsymbol{\lambda}(s,t)) + w}{\kappa_i(\boldsymbol{\lambda}(s,t)) - w}
$$

is an entire function of w and, in particular, does not contain negative powers of w in its expansion at $w = \infty$. On the other hand, we have

$$
\begin{aligned}
&\left(\left(w \frac{\partial \mathcal{S}_-}{\partial w}(w, \boldsymbol{\lambda}, s, t) + F_i(\boldsymbol{\lambda}) \right) \frac{\kappa_i(\boldsymbol{\lambda}) + w}{\kappa_i(\boldsymbol{\lambda}) - w} \right)_{<0} \\
=& w \frac{\partial \mathcal{S}_-}{\partial w}(w, \boldsymbol{\lambda}, s, t) \frac{\kappa_i(\boldsymbol{\lambda}) + w}{\kappa_i(\boldsymbol{\lambda}) - w} + \frac{2\kappa_i(\boldsymbol{\lambda}) F_i(\boldsymbol{\lambda})}{\kappa_i - w} = 0,
\end{aligned}
\tag{49}
$$

which follows from $\mathcal{S}_- = O(w^{-1})$ and (39). Therefore (44) holds, which, together with (43), leads to (42). Thus (18) is proved.

Equation (19) can be proved similarly. This completes the proof of Theorem 4.3.

5 Example

In this section we examine a simple example. Since we deal with the case $N = 1$, we omit the index of κ_i, λ_i, F_i and so on.

Let α be a complex parameter. Direct computation shows that the following function is a solution of the Löwner equation (4) for $N = 1$:

$$
\begin{aligned}
g(\lambda, z) &= -\alpha \frac{\sqrt{(1 - \lambda\alpha z^{-1})(1 - \lambda^{-1}\alpha z^{-1})} + (1 + \alpha z^{-1})}{\sqrt{(1 - \lambda\alpha z^{-1})(1 - \lambda^{-1}\alpha z^{-1})} - (1 + \alpha z^{-1})} \\
&= z \left(\frac{4\lambda}{(\lambda + 1)^2} - 2\frac{(\lambda - 1)^2}{(\lambda + 1)^2} \alpha z^{-1} + O(z^{-2}) \right).
\end{aligned}
\tag{50}
$$

The driving function $\kappa(\lambda)$ is a constant function: $\kappa(\lambda) \equiv \alpha$ and $\phi(\lambda)$ is determined by

$$
e^{-\phi} = \frac{4\lambda}{(\lambda + 1)^2}.
$$

If $|\alpha| = 1$ and λ is a real number greater than 1, $\lambda \geqq 1$, the above $g(\lambda, z)$ is a conformal mapping from

$$
\{z \in \mathbb{C} \mid |z| > 1\} \setminus \{t\alpha \mid 1 < t \leqq \lambda\}
$$

to the outside of the unit disk, $\{w \in \mathbb{C} \mid |w| > 1\}$.

The inverse function of $g(\lambda, z)$ is

$$
\begin{aligned}
f(\lambda, w) &= \alpha \frac{(1 + \lambda)(\alpha + w) + \sqrt{4\lambda(w - \alpha)^2 + (\lambda - 1)^2(\alpha + w)^2}}{(1 + \lambda)(\alpha + w) - \sqrt{4\lambda(w - \alpha)^2 + (\lambda - 1)^2(\alpha + w)^2}} \\
&= w \left(\frac{(\lambda + 1)^2}{4\lambda} + \frac{(\lambda - 1)^2}{2\lambda} \alpha w^{-1} + O(w^{-2}) \right).
\end{aligned}
\tag{51}
$$

Even in the case of $F(\lambda) = 0$, this example gives a non-trivial solution of the dcmKP hierarchy. As is seen from (51), the first Faber polynomial is

$$\Phi_1(\lambda) = \frac{(\lambda+1)^2}{4\lambda} w + \frac{(\lambda-1)^2}{2\lambda} \alpha. \tag{52}$$

Hence $v^1(\lambda) = \alpha(\lambda+1)^2/4\lambda$. If $t_n = 0$ for $n \geqq 2$, the hodograph relation (27) becomes

$$s + \alpha \frac{(\lambda+1)^2}{4\lambda} t_1 = 0. \tag{53}$$

Therefore

$$\frac{4\lambda}{(\lambda+1)^2} = -\frac{\alpha t_1}{s}, \qquad \frac{(\lambda-1)^2}{(\lambda+1)^2} = \frac{s+\alpha t_1}{s}.$$

Substituting them in (51), we have

$$\begin{aligned}
&\mathcal{L}(s,t)|_{t_n=0(n\geqq 2)} \\
&\quad = \alpha \frac{(\alpha+w)\sqrt{s} + \sqrt{-\alpha t_1 (w-\alpha)^2 + (s+\alpha t_1)(\alpha+w)^2}}{(\alpha+w)\sqrt{s} - \sqrt{-\alpha t_1 (w-\alpha)^2 + (s+\alpha t_1)(\alpha+w)^2}}.
\end{aligned} \tag{54}$$

6 Reduction to Löwner-Kufarev equation

In the theory of univalent functions, the Löwner equation (1) is generalized in various ways. (See, e.g., [D] §3.4.) Here we show that the *Löwner-Kufarev* equation (with multiple independent variables) of the type

$$\frac{\partial g}{\partial \lambda_i}(\boldsymbol{\lambda}, z) = g(\boldsymbol{\lambda}, z) \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \sum_{j=1}^{N} \mu_{ij}(\boldsymbol{\lambda}) \frac{\kappa_j(\boldsymbol{\lambda}) + g(\boldsymbol{\lambda}, z)}{\kappa_j(\boldsymbol{\lambda}) - g(\boldsymbol{\lambda}, z)} \tag{55}$$

turns into the Löwner equation (4) by means of suitable coordinate transformation. This gives another reduction of the dcmKP hierarchy when it is combined with the result in §4.

The argument in this section is rather formal and we assume genericity in many points.

We keep the assumption on the form of $g(\boldsymbol{\lambda}, z)$ (3) as in §2. Equation (55) is rewritten as the equation for the inverse function $f(\boldsymbol{\lambda}, w)$ (5) as follows:

$$\frac{\partial f}{\partial \lambda_i}(\boldsymbol{\lambda}; w) = \left(\sum_{j=1}^{N} \mu_{ij}(\boldsymbol{\lambda}) w \frac{w + \kappa_j(\boldsymbol{\lambda})}{w - \kappa_j(\boldsymbol{\lambda})} \right) \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial f}{\partial w}(\boldsymbol{\lambda}; w). \tag{56}$$

By comparing the coefficient of w^1 (i.e., the asymptotic behavior at $w = \infty$), it is shown that we need to assume

$$\sum_{j=1}^{N} \mu_{ij}(\boldsymbol{\lambda}) = 1. \tag{57}$$

Assume that f is an entire function of w. Since the left hand side of (56) thereby does not have singularities at $w = \kappa_j$ $(j = 1, \ldots, N)$, we have

$$\frac{\partial f}{\partial w}(\boldsymbol{\lambda}; \kappa_j(\boldsymbol{\lambda})) = 0 \tag{58}$$

for all j. Let us define new variables ζ_j as

$$\zeta_j = \zeta_j(\boldsymbol{\lambda}) := f(\boldsymbol{\lambda}; \kappa_j(\boldsymbol{\lambda})). \tag{59}$$

Assuming that the transformation

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \mapsto \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_N)$$

is invertible, we show that (56) becomes the Löwner equation (6) with respect to $\boldsymbol{\zeta}$.

By (58) and (56), we have

$$\begin{aligned}
\frac{\partial \zeta_j}{\partial \lambda_i} &= \left. \frac{\partial f}{\partial \lambda_i}(\boldsymbol{\lambda}; w) \right|_{w=\kappa_j(\boldsymbol{\lambda})} \\
&= \lim_{w \to \kappa_j(\boldsymbol{\lambda})} \left(\sum_{k=1}^{N} \mu_{ik}(\boldsymbol{\lambda}) w \frac{w + \kappa_k(\boldsymbol{\lambda})}{w - \kappa_k(\boldsymbol{\lambda})} \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial f}{\partial w}(\boldsymbol{\lambda}; w) \right) \\
&= \lim_{w \to \kappa_j(\boldsymbol{\lambda})} \left(\mu_{ij}(\boldsymbol{\lambda}) w \frac{w + \kappa_j(\boldsymbol{\lambda})}{w - \kappa_j(\boldsymbol{\lambda})} \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial f}{\partial w}(\boldsymbol{\lambda}; w) \right) \\
&= 2\mu_{ij}(\boldsymbol{\lambda}) \kappa_j(\boldsymbol{\lambda})^2 \frac{\partial \phi(\boldsymbol{\lambda})}{\partial \lambda_i} \frac{\partial^2 f}{\partial w^2}(\boldsymbol{\lambda}; \kappa_j(\boldsymbol{\lambda})).
\end{aligned}$$

We can rewrite this equation as

$$\mu_{ij}(\boldsymbol{\lambda}) \frac{\partial \phi}{\partial \lambda_i}(\boldsymbol{\lambda}) = \frac{\partial \zeta_j}{\partial \lambda_i} K_j^{-1}, \tag{60}$$

if

$$K_j := 2\kappa_j(\boldsymbol{\lambda})^2 \frac{\partial^2 f}{\partial w^2}(\boldsymbol{\lambda}; \kappa_j(\boldsymbol{\lambda})) \tag{61}$$

does not vanish. The Löwner-Kufarev equation (56) now turns into

the differential equation

$$
\begin{aligned}
\frac{\partial f}{\partial \zeta_i} &= \sum_{j=1}^{N} \frac{\partial \lambda_j}{\partial \zeta_i} \frac{\partial f}{\partial \lambda_j} = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial \lambda_j}{\partial \zeta_i} \mu_{jk} w \frac{w+\kappa_k}{w-\kappa_k} \frac{\partial \phi}{\partial \lambda_j} \frac{\partial f}{\partial w} \\
&= \sum_{k=1}^{N} \left(\sum_{j=1}^{N} \frac{\partial \lambda_j}{\partial \zeta_i} \frac{\partial \zeta_k}{\partial \lambda_j} K_k^{-1} \right) w \frac{w+\kappa_k}{w-\kappa_k} \frac{\partial f}{\partial w} \\
&= K_i^{-1} w \frac{w+\kappa_i}{w-\kappa_i} \frac{\partial f}{\partial w}
\end{aligned}
\tag{62}
$$

with respect to $\zeta_i' s$. This is almost (6) except for the overall coefficient. Actually, if we compare the asymptotic behavior at $w=\infty$, as we did when we derived (57), we see that

$$
K_i^{-1} = \frac{\partial \phi}{\partial \zeta_i}.
$$

Thus (56) is rewritten as

$$
\frac{\partial f}{\partial \zeta_i}(\boldsymbol{\zeta}; w) = w \frac{w+\kappa_i(\boldsymbol{\zeta})}{w-\kappa_i(\boldsymbol{\zeta})} \frac{\partial \phi(\boldsymbol{\zeta})}{\partial \zeta_i} \frac{\partial f}{\partial w}(\boldsymbol{\zeta}; w), \tag{63}
$$

which is nothing but the Löwner equation.

Combining the above result with Theorem 4.3, we have the following theorem.

Theorem 6.1. *Let $f(\boldsymbol{\lambda}, w)$ be a solution of the Löwner-Kufarev equation* (56) *of the form* (5) *and $\boldsymbol{\lambda}(s,t)$ be a solution of* (27). *Then the function $\mathcal{L} = \mathcal{L}(s,t;w)$ defined by*

$$
\mathcal{L}(s,t;w) := f(\boldsymbol{\lambda}(s,t), w)
$$

is a solution of the dcmKP hierarchy (13).

It is $\zeta_i(s,t)$ rather than $\lambda_i(s,t)$ that are the Riemann invariants of this case. The functions $\lambda_i(s,t)$ satisfy the equation

$$
\frac{\partial \lambda_i}{\partial t_n} = \sum_{j,k=1}^{N} \frac{\partial \lambda_i}{\partial \zeta_j} v_j^n(\boldsymbol{\lambda}(s,t)) \frac{\partial \zeta_j}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial s},
$$

namely,

$$
\frac{\partial \boldsymbol{\lambda}}{\partial t_n} = Z^{-1} V_n Z \frac{\partial \boldsymbol{\lambda}}{\partial s}, \qquad \boldsymbol{\lambda} = {}^t(\lambda_1, \ldots, \lambda_N), \tag{64}
$$

instead of (20). Here Z and V_n are $N \times N$ matrices defined by

$$
Z = \left(\frac{\partial \zeta_i}{\partial \lambda_j} \right)_{i,j=1,\ldots,N}, \qquad V_n = \mathrm{diag}(v_1^n, \ldots, v_N^n).
$$

Mañas, Martínez Alonso and Medina [MMAM] considered the chordal version (cf. (2)) of the above generalization. Our argument can be applicable to their case.

7 Concluding remarks

In this article we show that the dcmKP hierarchy can be reduced to the Löwner(-Kufarev) equations. Let us mention several further problems.

- The dcmKP hierarchy is a "half" of the dispersionless Toda lattice hierarchy. Can we construct the missing half by extending the radial Löwner equation? The dispersionless Toda equation has already appeared in [GMMA] but in a different context.
- The stochastic version of the Löwner equation and its application to the conformal field theories are intensively studied. See [LSW], [C] and references therein. It might be interesting if the "stochastic version" of dispersionless type integrable systems (if any!) can be applied to the conformal field theories.

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